Fourth solution. Note that

$$\left(\frac{x^2}{a}+\frac{y^2}{b}\right)\sqrt{\frac{\dfrac{b^2}{x^2}+\dfrac{a^2}{y^2}}{2}} \geq \sqrt{2(x^2+y^2)} \iff$$

$$\iff \left(\frac{x^2}{a}+\frac{y^2}{b}\right)^2\left(\frac{b^2}{x^2}+\frac{a^2}{y^2}\right) \geq 4(x^2+y^2) \iff$$

and we have

$$\left(\frac{x^2}{a}+\frac{y^2}{b}\right)^2\left(\frac{b^2}{x^2}+\frac{a^2}{y^2}\right) = \left(\frac{x^4}{a^2}+\frac{y^4}{b^2}+\frac{2x^2y^2}{ab}\right)\left(\frac{b^2}{x^2}+\frac{a^2}{y^2}\right) =$$

$$= \left(\frac{y^4}{x^2}+\frac{x^4}{y^2}\right)+\left(\frac{b^2x^2}{a^2}+\frac{2ax^2}{b}\right)+\left(\frac{a^2y^2}{b^2}+\frac{2by^2}{a}\right)$$

Since $\dfrac{u^2}{v} \geq 2u-v$ for any real u and any positive real v then

$$\frac{y^4}{x^2}+\frac{x^4}{y^2} \geq (2x^2-y^2)+(2y^2-x^2) = x^2+y^2.$$

or, by Cauchy Inequality

$$(x^2+y^2)\left(\frac{y^4}{x^2}+\frac{x^4}{y^2}\right) \geq \left(x\cdot\frac{y^2}{x}+y\cdot\frac{x^2}{y}\right)^2 \iff$$

$$\iff \frac{y^4}{x^2}+\frac{x^4}{y^2} \geq \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2$$

Also by AM-GM Inequality

$$\frac{b^2x^2}{a^2}+\frac{2ax^2}{b} = x^2\left(\frac{b^2}{a^2}+\frac{2a}{b}\right) \geq x^2\cdot 3\sqrt[3]{\frac{b^2}{a^2}\cdot\left(\frac{a}{b}\right)^2} = 3x^2,$$

$$\frac{a^2y^2}{b^2}+\frac{2by^2}{a} \geq 3y^2.$$

Hence,

$$\left(\frac{x^2}{a}+\frac{y^2}{b}\right)^2\left(\frac{b^2}{x^2}+\frac{a^2}{y^2}\right) \geq (x^2+y^2)+3x^2+3y^2.$$

Arkady Alt